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SYNTHETIC SEISMOGRAMS FOR DEEP SEISMIC SOUNDING STUDIES USING ASYMPTOTIC RAY THEORY

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ABSTRACT

In asymptotic ray theory, the solution for particle motion is assumed to be an infinite power series of inverse frequency and a vector amplitude, $\bar{A}_n(x, y, z)$, independent of frequency. A point source with any desired impulse response and radiation pattern is easily incorporated. A synthetic seismogram computer program has been written for a plane-layered homogeneous elastic media using the first or second terms of the expansion where necessary. Multiply converted refracted and reflected phases and also head waves at distances away from the critical angle are included. In addition, the phases are all identified and their amplitude-distance function plotted if desired. The synthetic seismograms are calculated for a model in southern Alberta and another in northwestern Ontario as obtained by deep seismic sounding programs. It is found that reflected phases dominate the seismograms and they are at least as important as head waves in the interpretation of experimental results.

INTRODUCTION

A geometrical theory of electromagnetic wave propagation has been generalized by Luneberg (1944) so that it is useful at both high and relatively low frequencies. The field quantities are developed as an asymptotic expansion in terms of powers of the reciprocal wave number. An essential feature of Luneberg's theory is that the flux of energy is confined to a ray tube, which is defined by trajectories orthogonal to the wave front, and cannot escape through the walls. Kline (1951) expressed the field quantities with a harmonic time dependence and an infinite power series in $1/(i\omega)$ combined with a space-dependent vector which is independent of frequency. The principal or frequency-independent term in the expansion gives results which are identical with those of geometrical ray theory. That is, the field quantities propagate along paths determined by Fermat's principle of least time. In addition to the kinematic quantities, the solution also has the amplitude and phase or dynamic characteristics for reflected and refracted rays. Higher order or frequency-dependent terms are of paramount importance when solving problems involving diffraction phenomena, curved interfaces and inhomogeneous media. As a result of its potential in yielding the kinematic and dynamic characteristics, which are asymptotic to the wave solution, this technique of expanding field quantities in a power series of inverse frequency is called an asymptotic ray theory.

The propagation of waves in an elastic media has been studied intensively with asymptotic ray theory by a group under the direction of Professor Petrashen at the University of Leningrad (Babich and Aleksiev, 1958; Skuridin and Gvozdev, 1958). Similar work was carried out on homogeneous and inhomogeneous media by Karal and Keller (1959). Various extensions involving diverse boundary conditions have been treated by Aleksiev, Babich and Gelchinsky (1961); Yanovskaya (1966); and Hron (1968) but these are relatively inaccessible. Therefore, a review of asymptotic

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ray theory, as applied in its simplest form to plane-layered homogeneous elastic solid and liquid media is presented in the first section.

A succeeding section discusses results obtained in programming algorithms partly developed in the theoretical section. Synthetic seismograms and amplitude-distance characteristics for selected phases are obtained for two models from an interpretation of deep seismic sounding programs in western Canada. The synthetic seismograms appear to be of great value to an interpreter of field records as they contain all reflected and converted phases which have a significant amplitude and also the head waves at distances from the critical point where the ray expansion is valid. The phases are identified automatically in the computer output and on its graphical presentation so that an interpretation of the seismogram is easily made.

Asymptotic Ray Theory

The equation of motion in an isotropic homogeneous medium is

$$\rho \frac{\partial^2 \bar{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \bar{u}) + \mu\nabla^2 \bar{u} \tag{1}$$

where $\bar{u}(x, y, z, t)$ is the particle displacement in a cartesian coordinate system; λ and μ are Lamé's constants and ρ is the density of the medium.

A solution for any wave equation near the wave front can be written generally as

$$\bar{u} = \bar{W}(x, y, z, t)s[t - \tau(x, y, z)] + \cdots \tag{2}$$

where s describes the source as a function of time and may be discontinuous at $t = 0$ and $\bar{W}$ is an analytic function in the neighborhood of the wave front. The equation describing the position of the wave front at time t is

$$t = \tau(x, y, z) \tag{3}$$

where τ is called a phase function and depends upon the ray path from the source to observer at $M(x, y, z)$. Let

$$\Delta\tau = t - \tau \tag{4}$$

be the time a short distance from the wave front and expand $\bar{W}$ in a Taylor series about $t = \tau + \Delta\tau$. Equation (2) is then

$$\bar{u} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n \bar{W}}{\partial t^n}\right]_{t=\tau(x,y,z)} \cdot (\Delta\tau)^n s\{t - \tau(x, y, z)\}. \tag{5}$$

Note that the expansion now consists of a spatial term in square brackets separated from a simple term that is both time and spatially dependent. A monochromatic sinusoidal solution may also be written as

$$\bar{u}(x, y, z, t) = \bar{\mathfrak{A}}(x, y, z, \omega)e^{i\omega(t-\tau)}. \tag{6}$$

Expanding $\bar{\mathfrak{A}}$ as a power series in $1/(i\omega)$ and coefficients, $\bar{A}_n$, which are independent of frequency we have

$$\bar{u}(x, y, z, t) = \sum_{n=0}^{\infty} \bar{A}_n(x, y, z) \frac{e^{i\omega(t-\tau)}}{(i\omega)^n} . \tag{7}$$

A particular case for harmonic waves is when s in equation (5) is

$$\frac{(\Delta\tau)^n}{n!} s(t - \tau) = \frac{e^{i\omega(t-\tau)}}{(i\omega)^n} \equiv s_n(t - \tau). \tag{8}$$

Equation (7) is in a useful form because the vector $\bar{A}_n$ is independent of frequency. The frequency-dependent portion of both the source function and the wave modification due to the medium is contained in the term $s_n[t - \tau(x, y, z)]$. Thus equation (7) can be written as

$$\bar{u}(x, y, z, t) = \sum_{n=0}^{\infty} \bar{A}_n(x, y, z) s_n[t - \tau(x, y, z)]. \tag{9}$$

Equation (9) can be generalized to an impulsive source function by first obtaining the Fourier transform of the desired impulse response, $s(t)$

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} \, dt. \tag{10}$$

By summing (9) over all frequencies, the displacement at any location is given by

$$\bar{u}(x, y, z, t) = \sum_{n=0}^{\infty} \bar{A}_n(x, y, z) \frac{1}{\pi} R \int_{\omega_0}^{\infty} \frac{S(\omega)}{(i\omega)^n} e^{i\omega(t-\tau)} \, d\omega. \tag{11}$$

To avoid difficulties as the frequency approaches zero, the function $S(\omega)$ is defined to be zero from $\omega = 0$ to ω_0 in (11). Thus we can let s_n in (9) represent a modified impulse response function defined by the inverse Fourier transform

$$s_n(t - \tau) = \frac{1}{\pi} R \int_{\omega_0}^{\infty} \frac{S(\omega)}{(i\omega)^n} e^{i\omega(t-\tau)} \, d\omega. \tag{12}$$

Note that s_n has the following important property

$$s_n'(t - \tau) = s_{n-1}(t - \tau) \tag{13}$$

where the prime indicates differentiation with respect to the argument. Every higher order term, s_n, can be obtained by integration of s_{n-1}. It is also seen that for $n = 1$ or 2 in equation (11) terms like $e^{i\omega\Delta\tau}/i\omega$ and $-e^{i\omega\Delta\tau}/\omega^2$ will approach zero rapidly in the high-frequency limit as $\omega \to \infty$. Therefore, it is often necessary to take only the leading term ($n = 0$) or the second one ($n = 1$) to obtain a useful asymptotic solution.

In substituting equation (9) into the elastic-wave equation (1), the following quantities are necessary

$$\frac{\partial^2 \bar{u}}{\partial t^2} = \sum_{n=0}^{\infty} \bar{A}_n s_{n-2} \tag{14}$$

$$\nabla\cdot\bar{u} = \sum_{n=0}^{\infty} \{-s_{n-1}\,\text{grad}\,\tau\cdot\bar{A}_n + s_n\nabla\cdot\bar{A}_n\} \tag{15}$$

$$\nabla(\nabla\cdot\bar{u}) = \sum_{n=0}^{\infty} \{s_{n-2}(\text{grad}\,\tau\cdot\bar{A}_n)\,\text{grad}\,\tau - s_{n-1}\,\text{grad}\,(\text{grad}\,\tau\cdot\bar{A}_n)$$

$$- s_{n-1}(\nabla\cdot\bar{A}_n)\,\text{grad}\,\tau + s_n\,\text{grad}\,(\nabla\cdot\bar{A}_n)\} \tag{16}$$

$$\nabla^2\bar{u} = \sum_{n=0}^{\infty} \left\{ s_{n-2}\bar{A}_n(\text{grad}\,\tau)^2 - \sum_{j=1}^{3} \bar{\epsilon}_j 2s_{n-1}\,\text{grad}\,\tau\cdot\text{grad}\,A_{nj}\right.$$

$$\left. - s_{n-1}\bar{A}_n\nabla^2\tau + s_n\nabla^2\bar{A}_n \right\} \tag{17}$$

where

$$\bar{\epsilon}_1 = (\bar{\imath}, 0, 0); \quad \bar{\epsilon}_2 = (0, \bar{\jmath}, 0); \quad \bar{\epsilon}_3 = (0, 0, \bar{k}) \tag{18}$$

are unit vectors in the direction of the coordinate axes. The following three operators are formed by collecting terms in s_{n-2}, s_{n-1}, and s_n from equations (14), (16), and (17), respectively

$$\bar{N}(\tau, \bar{A}_n) = (\lambda + \mu)(\text{grad}\,\tau\cdot\bar{A}_n)\,\text{grad}\,\tau + [\mu(\text{grad}\,\tau)^2 - \rho]\bar{A}_n \tag{19}$$

$$\bar{M}(\tau, \bar{A}_n) = (\lambda + \mu)[(\nabla\cdot\bar{A}_n)\,\text{grad}\,\tau + \text{grad}\,(\bar{A}_n\cdot\text{grad}\,\tau)]$$

$$+ \mu[(\nabla^2\tau)\bar{A}_n + \sum_{j=1}^{3} \bar{\epsilon}_j\, 2\,\text{grad}\,\tau\cdot\text{grad}\,A_{nj}] \tag{20}$$

$$\bar{L}(\bar{A}_n) = (\lambda + \mu)\,\text{grad}\,(\nabla\cdot\bar{A}_n) + \mu\nabla^2\bar{A}_n\,. \tag{21}$$

The wave equation for small elastic displacement, (1), is then

$$\sum_{n=0}^{\infty} [s_{n-2}\bar{N}(\tau, \bar{A}_n) - s_{n-1}\bar{M}(\tau, \bar{A}_n) + s_n\bar{L}(\bar{A}_n)] = 0. \tag{22}$$

Letting

$$A_{-2} = A_{-1} \equiv 0 \tag{23}$$

equation (22) becomes

$$\sum_{n=-2}^{\infty} s_n(t - \tau)[\bar{N}(\tau, \bar{A}_{n+2}) - \bar{M}(\tau, \bar{A}_{n+1}) + \bar{L}(\bar{A}_n)] = 0. \tag{24}$$

Each power in ω must be equal to zero independently for (24) to be generally valid. Therefore,

$$\bar{N}(\tau, \bar{A}_{n+2}) = \bar{M}(\tau, \bar{A}_{n+1}) - \bar{L}(\bar{A}_n) \qquad n = -2, -1, 0, 1, 2, \cdots. \tag{25}$$

Setting $n = -2$ yields an operator equation only in $\bar{N}$

$$\bar{N}(\tau, \bar{A}_0) = 0. \tag{26}$$

Writing out this relation in terms of the components and letting

$$G = \frac{\mu(\text{grad } \tau)^2 - \rho}{\lambda + \mu} \tag{27}$$

$$\tau_i = \frac{\partial \tau}{\partial x_i} \qquad i = 1, 2, 3 \tag{28}$$

we have

$$(\lambda + \mu)^3 \begin{pmatrix} \tau_1^2 + G & \tau_1\tau_2 & \tau_1\tau_3 \\ \tau_2\tau_1 & \tau_2^2 + G & \tau_2\tau_3 \\ \tau_3\tau_1 & \tau_3\tau_2 & \tau_3^2 + G \end{pmatrix} \begin{pmatrix} A_{01} \\ A_{02} \\ A_{03} \end{pmatrix} = 0. \tag{29}$$

If $\bar{A}_0$ is not to be identically zero, then the determinant of the matrix must vanish. After some algebra we find it to be

$$[\mu(\text{grad } \tau)^2 - \rho]^2[(\lambda + 2\mu)(\text{grad } \tau)^2 - \rho] = 0. \tag{30}$$

Setting the first term in square brackets equal to zero gives the eikonal equation for shear or S waves

$$(\text{grad } \tau_\beta)^2 = \rho/\mu = 1/\beta^2 \tag{31}$$

where β is the velocity of shear waves traveling along rays in the direction grad τ_β or orthogonal to the surface of constant phase (τ_β = constant). Setting the second term in (30) equal to zero yields the eikonal equation for compressional or P waves.

$$(\text{grad } \tau_\alpha)^2 = \frac{\rho}{\lambda + 2\mu} = \frac{1}{\alpha^2} \tag{32}$$

where α is the velocity of waves traveling along rays orthogonal to another surface of constant phase (τ_α = constant).

From (32) the ray path is found by requiring that the element of distance, ds, is taken in such a way that the travel time, τ, from M_0 to M is a minimum

$$\tau(x, y, z) = \tau_0 + \text{min.} \int_{M_0}^{M} \frac{ds}{\alpha(x, y, z)}. \tag{33}$$

In a homogeneous isotropic medium where α is a constant, τ is a linear function of the distance from M_0 to M

$$\tau = \tau_0 + \frac{M - M_0}{\alpha}. \tag{34}$$

Consider a central ray described in a spherical coordinate system (r, θ, ϕ) where θ is the longitudinal angle and ϕ is the colatitude (see Figure 1). The source function may have an angular dependence $C_n(\theta, \phi)$ which is prescribed as an initial condition on a unit sphere surrounding the source. Describe a ray tube about a central ray specified by (θ_0, ϕ_0) with a cross section determined by $\theta_0 \pm \frac{1}{2} d\theta_0$ and $\phi_0 \pm \frac{1}{2} d\phi_0$. The area on the surface r = constant is given by

$$dA = J\, d\theta_0\, d\phi_0 \tag{35}$$

where

$$J = \left| r\hat{u}_\phi \times r \sin\theta \hat{u}_\theta \right| \tag{36}$$

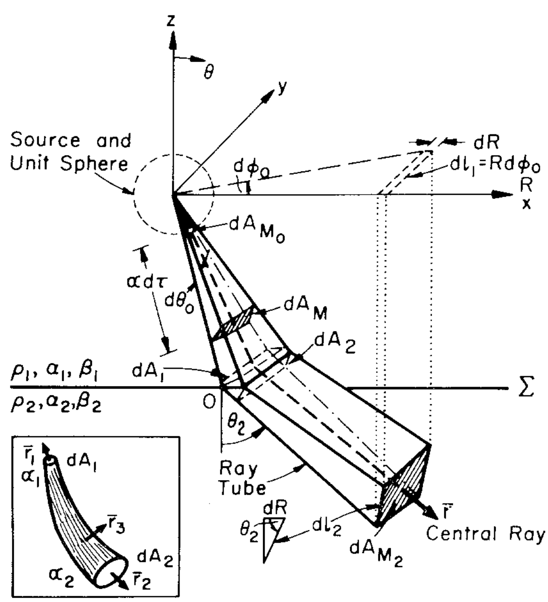

FIG. 1. The ray tube before and after refraction at a plane interface. Either cartesian (x, y, z), spherical (r, θ, ϕ), or circular cylindrical (r, ϕ, z) coordinates will be used depending upon the problem considered. The *small insert* shows the unit vectors on a small segment of the ray tube as used in Appendix 1.

with $\hat{u}_\phi$ and $\hat{u}_\theta$ being unit vectors. J is proportional to the cross-section area of the ray tube at distance r. The volume of the ray tube from M_0 to M in Figure 1 is

$$dV = J\, d\theta\, d\phi \alpha\, d\tau_\alpha . \tag{37}$$

To determine the nature of the waves, substitute equation (32) for ρ into the vector operator $\bar{N}(\tau, \bar{A}_0)$ of equation (26).

$$(\lambda + \mu)(\text{grad } \tau_\alpha \cdot \bar{A}_0) \text{ grad } \tau_\alpha + [\mu(\text{grad } \tau_\alpha)^2 - (\lambda + 2\mu)(\text{grad } \tau_\alpha)^2]\bar{A}_0 = 0$$

or

$$(\lambda + \mu)[(\text{grad } \tau_\alpha \cdot \bar{A}_0) \text{ grad } \tau_\alpha - (\text{grad } \tau_\alpha)^2 \bar{A}_0] = 0. \tag{38}$$

Dividing both sides by $(\lambda + \mu)(\text{grad } \tau_\alpha)^2 = (\lambda + \mu)/\alpha^2$ gives

$$\bar{A}_0 = [\alpha^2 \text{ grad } \tau_\alpha \cdot \bar{A}_0] \text{ grad } \tau_\alpha$$

or

$$\bar{A}_0 \equiv \bar{P}_0 = P_0(x, y, z) \text{ grad } \tau_\alpha . \tag{39}$$

Therefore, a wave moving with velocity α has particle motion in the direction of the ray and the amplitude will be identified as P_0. Similarly, if we let $\rho = \mu \ (\text{grad } \tau_\beta)^2$ in (26) we have

$$(\lambda + \mu)(\text{grad } \tau_\beta \cdot \bar{A}_0) \text{ grad } \tau_\beta + [\mu(\text{grad } \tau_\beta)^2 - \mu(\text{grad } \tau_\beta)^2]\bar{A}_0 = 0. \tag{40}$$

Dividing both sides by $(\lambda + \mu)$ grad τ_β and renaming the vector field as $\bar{S}_0$ instead of $\bar{A}_0$ gives

$$\text{grad } \tau_\beta \cdot \bar{S}_0 = 0. \tag{41}$$

Therefore, the phase for $n = 0$ propagating with velocity β has particle motion perpendicular to the ray trajectory. Vector operator $\bar{N}(\tau, \bar{A}_0)$ is seen to yield the eikonal equations for P and S waves and also specifies the direction of particle displacement for the principal term $(n = 0)$ in the ray series.

To solve for the next term in the expansion for compressional waves $(\bar{A} \equiv \bar{P})$, let $n = -1$ in equation (25).

$$\bar{N}(\tau_\alpha, \bar{P}_1) = \bar{M}(\tau_\alpha, P_0 \text{ grad } \tau_\alpha). \tag{42}$$

Take a projection of the vector $\bar{N}$ in the direction of the ray, grad τ_α.

$$\begin{aligned} \bar{N} \cdot \text{grad } \tau_\alpha &= (\lambda + \mu)(\text{grad } \tau_\alpha \cdot \bar{P}_1)(\text{grad } \tau_\alpha)^2 \\ &\qquad + [\mu(\text{grad } \tau_\alpha)^2 - \rho]\bar{P}_1 \text{ grad } \tau_\alpha \\ &= [(\lambda + 2\mu)(\text{grad } \tau_\alpha)^2 - \rho]\bar{P}_1 \cdot \text{grad } \tau_\alpha = 0 \end{aligned} \tag{43}$$

since $(\text{grad } \tau_\alpha)^2 = 1/\alpha^2$ and $\lambda + 2\mu = \rho\alpha^2$. Therefore, by (42) and (43) both a projection of $\bar{M}$ and $\bar{N}$ in the direction of grad τ_α is zero and, thus, both vector operators are perpendicular to the ray direction.

$$\bar{M}(\tau_\alpha, P_0 \text{ grad } \tau_\alpha) \cdot \text{grad } \tau_\alpha = 0. \tag{44}$$

Equation (42) can be considered an eigenvalue problem in the unknown components of $\bar{P}_1$. Since grad τ_α is known, P_0 can be solved from (44) using a relation derived in the appendix equation (A9) for the left-hand side.

$$\bar{M}(\tau_\alpha, P_n \text{ grad } \tau_\alpha) \cdot \text{grad } \tau_\alpha = \frac{2\rho}{\alpha^2 J^{1/2}} \frac{d}{d\tau_\alpha} (P_n J^{1/2}) \tag{45}$$

or for $n = 0$

$$\frac{d}{d\tau_\alpha}(P_0 J^{1/2}) = 0. \tag{46}$$

Integration of this expression yields a constant, $C_0(\theta, \phi)$ which is determined by the initial conditions on the amplitude of compressional waves as a function of angle on a unit sphere surrounding the source

$$P_0 \sqrt{J} = C_0(\theta, \phi)$$

or from (39)

$$\bar{P}_0 = \frac{1}{\sqrt{J}} C_0(\theta, \phi) \text{ grad } \tau_\alpha . \tag{47}$$

The quantities $\bar{P}_1, \bar{P}_2, \cdots$ can be resolved into three components along perpendicular directions

$$\bar{P}_n = P_n \frac{\bar{r}}{\alpha} + p_n \bar{n} + p_{nb} \bar{b} \qquad n \geq 1 \tag{48}$$

where

$$\frac{\bar{r}}{\alpha} = \text{grad } \tau_\alpha \tag{49}$$

is a vector perpendicular to the wave front as seen by equation (32); $\bar{n}$ and $\bar{b}$ are unit vectors in the direction of the principal normal and the binormal, respectively, and both are orthogonal to $\bar{r}$

$$\bar{n} \cdot \text{grad } \tau_\alpha = \bar{b} \cdot \text{grad } \tau_\alpha = 0 \tag{50}$$

$$\bar{b} = \text{grad } \tau_\alpha \times \bar{n}. \tag{51}$$

Substituting (48) into (42) and taking projections in either the $\bar{n}$ or $\bar{b}$ direction yields solutions for the transverse components of $\bar{P}_1$

$$p_1 = \frac{-\alpha^2}{\rho(\alpha^2 - \beta^2)} [\bar{M}(\tau_\alpha, \bar{P}_0)] \cdot \bar{n} \tag{52}$$

$$p_{1b} = \frac{-\alpha^2}{\rho(\alpha^2 - \beta^2)} [\bar{M}(\tau_\alpha, \bar{P}_0)] \cdot \bar{b}. \tag{53}$$

Other terms in the ray series are obtained by setting $n = 0, 1, \cdots$ successively in equation (25). As in equation (42) this is an eigenvalue problem with the necessary and sufficient conditions for the existence of a solution being equations similar to (44)

$$[\bar{M}(\tau_\alpha, \bar{P}_n) - \bar{L}(\bar{P}_{n-1})] \cdot \text{grad } \tau_\alpha = 0 \qquad n \geqq 1. \tag{54}$$

From this expression, one can solve for the components of P_n using equation (45) for $\bar{M}(\tau_\alpha, P_n \text{ grad } \tau_\alpha) \cdot \text{grad } \tau_\alpha$

$$P_n = \frac{C_n(\theta, \phi)}{J^{1/2}} + \frac{\alpha^2}{2\rho J^{1/2}} \int_0^{\tau_\alpha} J^{1/2}[\bar{L}(\bar{P}_{n-1}) - \bar{M}(\tau_\alpha, \bar{p}_n)] \cdot \text{grad } \tau_\alpha \, d\tau_\alpha \qquad n = 1, 2, \cdots \tag{55}$$

where $C_n(\theta, \phi)$ is a constant of integration for the amplitude of compressional waves as a function of θ and ϕ on the unit sphere surrounding the source. The components perpendicular to the ray trajectory, which are necessary for head waves, are obtained by substituting (48) into (25)

$$\bar{p}_n = \frac{-\alpha^2}{\rho(\alpha^2 - \beta^2)} [\bar{M}(\tau_\alpha, \bar{P}_{n-1}) - \bar{L}(p_{n-2})] \qquad n \geq 1. \tag{56}$$

The coefficients for S waves will be defined similarly to those for compressional waves

$$\bar{S}_0 = s_0 \bar{n} + s_{0b} \bar{b} \tag{57}$$

$$\bar{S}_n = S_n \frac{\bar{r}}{\beta} + s_n \bar{n} + s_{nb} \bar{b} \qquad n \geqq 1 \tag{58}$$

where now, $\bar{r}$ is a unit vector in the direction of grad τ_β. Substituting S_n for $\bar{A}$ in (19), it is easily found that

$$\bar{N}(\tau_\beta, \bar{S}_n) = \frac{\alpha^2 - \beta^2}{\beta^2} \rho S_n \text{ grad } \tau_\beta. \tag{59}$$

Then with $n = -1, \cdots$ successively and equation (59), we have

$$S_1 \text{ grad } \tau_\beta = \frac{\beta^2}{\rho(\alpha^2 - \beta^2)} \bar{M}(\tau_\beta, \bar{S}_0) \tag{60}$$

$$S_n \text{ grad } \tau_\beta = \frac{\beta^2}{\rho(\alpha^2 - \beta^2)} [\bar{M}(\tau_\beta, \bar{S}_{n-1}) - \bar{L}(\bar{S}_{n-2})] \quad n = 2, 3, \cdots. \tag{61}$$

The components perpendicular to the ray are found by multiplying (60) and (61) by $\bar{n}$ and $\bar{b}$

$$\bar{M}(\tau_\beta, \bar{S}_0) \cdot \bar{n} = 0 \tag{62}$$

$$\bar{M}(\tau_\beta, \bar{S}_0) \cdot \bar{b} = 0 \tag{63}$$

$$[\bar{M}(\tau_\beta, \bar{S}_n) - \bar{L}(\bar{S}_{n-1})] \cdot \bar{n} = 0 \qquad n \geqq 1 \tag{64}$$

$$[\bar{M}(\tau_\beta, \bar{S}_n) - \bar{L}(\bar{S}_{n-1})] \cdot \bar{b} = 0 \qquad n \geqq 1. \tag{65}$$

Having obtained s_n and s_{nb} from the four equations above, S_n ($n = 1, 2, \cdots$) may be found from (60) or (61).

Where there is a discontinuity in the parameters of the media, or their derivatives, a boundary surface, Σ, is introduced. The rays may be continued away from the boundary by having the ray series satisfy a set of boundary conditions. Typically, the media on either side of the surface are in welded contact, and we require continuity of displacement and stress. The boundary conditions are stated in terms of the functions as they occur when the wave front impinges on the interface. From (9) the displacements of the rays arriving or leaving a boundary are specified by

$$\bar{u}_\nu(x, y, z, t) = \sum_{n=0}^{\infty} \bar{A}_{n\nu}(x, y, z) s_n(t - \tau) \tag{66}$$

where the index ν specifies the phase

$$\begin{aligned}
\nu &= 0, && \text{incident wave} \\
\nu &= 1, && \text{reflected } P \text{ wave } (\bar{A}_{n1} \equiv \bar{P}_{n1}) \\
\nu &= 2, && \text{refracted } P \text{ wave } (\bar{A}_{n2} \equiv \bar{P}_{n2}) \\
\nu &= 3, && \text{reflected } S \text{ wave } (\bar{A}_{n3} \equiv \bar{S}_{n3}) \\
\nu &= 4, && \text{refracted } S \text{ wave } (\bar{A}_{n4} \equiv \bar{S}_{n4}).
\end{aligned} \tag{67}$$

The incident wave will be assumed to be a P wave $(\bar{A}_{n0} \equiv \bar{P}_{n0})$ and the convention for vector direction is given in Figure 2. A local coordinate system is introduced with the z axes normal to the surface. Shear waves that are horizontally polarized (SH) will not be discussed here because they form an independent system for which equations are easily derived but seldom used in practice. The relevant equations for normal and shear stress are

$$p_{zz} = \lambda \nabla \cdot \bar{u} + 2\mu \frac{\partial u_z}{\partial z} \tag{68}$$

$$p_{xz} = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right). \tag{69}$$

The phase function, τ, is determined by solving equations (31) and (32) while equation (3) requires that the phase, τ, be continuous on the boundary.

$$\tau_\nu(x, y, z) = \tau_0(x, y, z) \qquad \nu = 1, \cdots 4. \tag{70}$$

From these conditions Snell's Law is required

$$\frac{\partial \tau_\nu}{\partial x} = \frac{\partial \tau_0}{\partial x} \tag{71}$$

or

$$\frac{\sin \theta_\nu}{c_\nu} = \frac{\sin \theta_0}{c_0} \begin{cases} \nu = 1, 2, 3, 4 \\ c_0 = \alpha_1 ; c_1 = \alpha_1 ; c_2 = \alpha_2 \\ c_3 = \beta_1 ; c_4 = \beta_2 \end{cases} \tag{72}$$

$$\left\{\frac{\partial \tau_\nu}{\partial y} = \frac{\partial \tau_0}{\partial y} = 0\right\}_{y=0} \tag{73}$$

$$\frac{\partial \tau_\nu}{\partial z} = (-1)^{\nu+1} \frac{\cos \theta_\nu}{c_\nu}. \tag{74}$$

Angles θ_ν are acute angles between the z axes and the vector $\bar{r}_\nu$ for the ν phase. If

$$\frac{\partial \tau_\nu}{\partial x} > \frac{1}{c_\nu} \tag{75}$$

the solution for θ_ν and τ_ν is complex. Both τ_ν and its complex conjugate, $\tau_\nu{}^*$, satisfy equations (31), (32) and (70). The choice is determined by the physical condition that the solution for s_n in (12) approach zero as the wave front moves away from the boundary. Therefore, the choice between τ_ν and $\tau_\nu{}^*$ is made on the basis that the

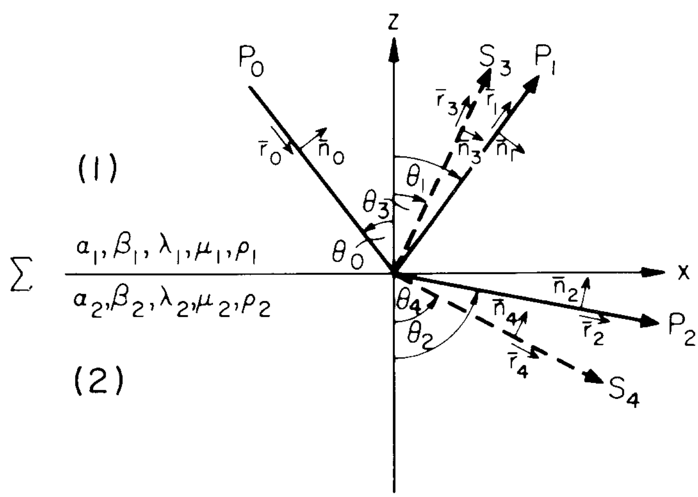

FIG. 2. Rays generated at a boundary and the convention used for defining angles and unit vectors.

imaginary part must have a negative sign. Such a ray with a complex phase function indicates that we have encountered an inhomogeneous wave, otherwise, it is known as an "evanescent" wave, that is, one with an exponentially decaying term.

With the aid of equations (66) to (74) and after considerable algebra, the continuity equations may be written in the following manner. For convenience let

$$(\alpha_i, \beta_i, \lambda_i, \mu_i, \rho_i)_{i=0,3} = (\alpha_1, \beta_1, \lambda_1, \mu_1, \rho_1)$$

$$(\alpha_4, \beta_4, \lambda_4, \mu_4, \rho_4) = (\alpha_2, \beta_2, \lambda_2, \mu_2, \rho_2).$$

Continuity of the x component of particle motion is

$$\sum_{\nu=1}^{2} (-1)^\nu \sin \theta_\nu P_{n\nu} + \sum_{\nu=3}^{4} (-1)^\nu s_{n\nu} \cos \theta_\nu$$

$$= P_{n0} \sin \theta_0 - \sum_{\nu=3}^{4} (-1)^\nu S_{n\nu} \sin \theta_\nu - \sum_{\nu=0}^{2} (-1)^{\delta_{\nu 0}+\nu} p_{n\nu} \cos \theta_\nu \tag{76}$$

Continuity of the z component of particle motion is

$$\sum_{\nu=1}^{2} P_{n\nu} \cos \theta_\nu - \sum_{\nu=3}^{4} s_{n\nu} \sin \theta_\nu$$
$$= P_{n0} \cos \theta_0 - \sum_{\nu=3}^{4} S_{n\nu} \cos \theta_\nu + \sum_{\nu=0}^{2} (-1)^{\delta_{\nu 0}} p_{n\nu} \sin \theta_\nu . \quad (77)$$

The equations of stress contain powers of frequency and in deriving the following expressions we have used the requirement that each term in n must be independent. Continuity of normal stress, p_{zz}, is

$$\sum_{\nu=1}^{2} (-1)^\nu \frac{\lambda_\nu + 2\mu_\nu \cos^2 \theta_\nu}{\alpha_\nu} P_{n\nu} - \sum_{\nu=3}^{4} (-1)^\nu \frac{\mu_\nu}{\beta_\nu} s_{n\nu} \sin 2\theta_\nu$$
$$= \sum_{\nu=0}^{2} (-1)^{\nu+\delta_{\nu 0}} \left[\lambda_\nu \nabla \cdot \bar{P}_{n-1,\nu} + 2\mu_\nu \frac{\partial (P_z)_{n-1,\nu}}{\partial z} + \mu_\nu p_{n\nu} \frac{\sin 2\theta_\nu}{\alpha_\nu} \right]$$
$$+ \frac{\lambda_0 + 2\mu_0 \cos^2 \theta_0}{\alpha_0} P_{n0} + \sum_{\nu=3}^{4} (-1)^\nu$$
$$\cdot \left[\lambda_\nu \nabla \cdot \bar{S}_{n-1,\nu} + 2\mu_\nu \frac{\partial (S_z)_{n-1,\nu}}{\partial z} - \frac{\lambda_\nu + 2\mu_\nu \cos^2 \theta_\nu}{\beta_\nu} S_{n\nu} \right]. \quad (78)$$

Continuity of shear stress, p_{zx} is

$$\sum_{\nu=1}^{2} \mu_\nu P_{n\nu} \frac{\sin 2\theta_\nu}{\alpha_\nu} + \sum_{\nu=3}^{4} \frac{\mu_\nu}{\beta_\nu} s_{n\nu} \cos 2\theta_\nu = \mu_0 P_{n0} \frac{\sin 2\theta_0}{\alpha_0}$$
$$+ \sum_{\nu=0}^{2} \left\{ (-1)^{1+\delta_{\nu 0}} \mu_\nu p_{n\nu} \frac{\cos 2\theta_\nu}{\alpha_\nu} - (-1)^{\nu+\delta_{\nu 0}} \mu_\nu \left[\frac{\partial (P_x)_{n-1,\nu}}{\partial z} + \frac{\partial (P_z)_{n-1,\nu}}{\partial x} \right] \right\}$$
$$- \sum_{\nu=3}^{4} \left\{ \mu_\nu S_{n\nu} \frac{\sin 2\theta_\nu}{\beta_\nu} + (-1)^\nu \mu_\nu \left[\frac{\partial (S_x)_{n-1,\nu}}{\partial z} + \frac{\partial (S_z)_{n-1,\nu}}{\partial x} \right] \right\}. \quad (79)$$

Note that the component S_x, S_z, P_x, P_z refer to component of vector $\bar{S}$ and $\bar{P}$. The determinant of the system of equations above is not zero and a solution is obtainable in the following form by setting $n = 0$

$$\frac{P_{0\nu}}{P_{00}} = K_\nu \qquad \nu = 1, 2 \quad (80)$$

$$\frac{s_{0\nu}}{P_{00}} = k_\nu \qquad \nu = 3, 4 \quad (81)$$

where K_ν and k_ν are complex coefficients of reflection ($\nu = 1, 3$) or refraction ($\nu = 2, 4$) for P and S waves, respectively. These are essentially identical to coefficients calculated by Knott (1899) for reflection and refraction of plane waves at an interface and, therefore, we need not consider them further in this discussion.

Finally, the conservation of energy must be applied as we have assumed that energy flux does not escape through the sides of the ray tube. Thus, the power or energy flux is constant on each cross section which cuts a wave tube unless there is an interface. Let

$dA\,(M_0)$ be the cross-section area of the ray tube on the unit sphere at the source and let $dA\,(M)$ be the cross section at some other point, M, on the ray. Then

$$\rho_{M_0} c_{M_0} \bar{\dot{u}}_{M_0}^2 \, dA\,(M_0) = \rho_M c_M \bar{\dot{u}}_M{}^2 \, dA\,(M) \tag{82}$$

where $\dot{u}$ is the particle velocity and c is the phase velocity of the wave in the medium. Integration gives the displacement at M.

$$\bar{u}(M) = \sqrt{\frac{(\rho c)_{M_0}}{(\rho c)_M}} \frac{\bar{u}(M_0)}{L} \tag{83}$$

where

$$L = \sqrt{\frac{J_M}{J_{M_0}}} = \sqrt{\frac{dA(M)}{dA(M_0)}} \tag{84}$$

is the geometric expansion factor of the ray tube, otherwise known as the square root of the Jacobian of the mapping of the wave front at M_0 to the wave front at M by the ray (θ_0, ϕ_0). With conservation of energy the amplitude of equation (47) becomes

$$P_{00} = \sqrt{\frac{(\rho c)_{M_0}}{(\rho c)_M}} \frac{c_0(\theta_0, \phi_0)}{L}. \tag{85}$$

After a ray is refracted or reflected, it is more convenient to use a circular cylindrical system (R, ϕ, z) as in Figure 1. By (80), the amplitude of the refracted wave, P_{02}, is given by

$$P_{02}^{(2)} = K_2 P_{00}^{(1)} \tag{86}$$

where the superscript denotes the layer being considered when there may be some ambiguity. At point 0, just above the first interface, the amplitude of the incident wave is given by (83)

$$P_{00}^{(1)}(0) = \sqrt{\frac{dA(M_0)}{dA_1}}\, P_{00}^{(1)}(M_0). \tag{87}$$

If the amplitude at 0 just below the interface is known, then, the amplitude of P or S refracted waves at point M_2 is

$$\begin{Bmatrix} P_{02}^{(2)} \\ s_{04}^{(2)} \end{Bmatrix}_{M_2} = \sqrt{\frac{dA_2}{dA(M_2)}} \begin{Bmatrix} P_{02}^{(2)} \\ s_{04}^{(2)} \end{Bmatrix}_0 . \tag{88}$$

Using (87) to relate the amplitudes across the boundary we have

$$\begin{Bmatrix} P_{02} \\ s_{04} \end{Bmatrix}_{M_2} = \sqrt{\frac{dA_2}{dA(M_2)}} \begin{Bmatrix} K_2 \\ k_4 \end{Bmatrix} \sqrt{\frac{dA(M_0)}{dA_1}}\, P_{00}(M_0). \tag{89}$$

Note that by (83) the densities and velocities at M_0, 0 and M_2 would need to be incorporated in the square-root terms if the medium were inhomogeneous. By geometry

$dA_1 = ds \cos \theta_0$ and $dA_2 = ds \cos \theta_2$ so we can write (89) as

$$\begin{Bmatrix} P_{02} \\ s_{04} \end{Bmatrix}_{M_2} = \begin{Bmatrix} K_2 \\ k_4 \end{Bmatrix} \frac{P_{00}(M_0)}{L} \tag{90}$$

where

$$L = \sqrt{\frac{dA(M_2)}{dA(M_0)}} \sqrt{\frac{\cos \theta_0}{\cos \theta_\nu}} \qquad \nu = 1, 2, 3, 4. \tag{91}$$

For reflected waves at location M_1 the amplitude is found analogously as

$$\begin{Bmatrix} P_{01} \\ s_{03} \end{Bmatrix}_{M_1} = \begin{Bmatrix} K_1 \\ k_3 \end{Bmatrix} \frac{P_{00}(M_0)}{L} . \tag{92}$$

On the unit sphere, the cross-section area of the ray tube, according to (35) is

$$dA\,(M_0) = \sin \theta_0 \, d\theta_0 \, d\phi_0 . \tag{93}$$

From Figure 1,

$$dA(M_2) = dl_1 \cdot dl_2 = (R\, d\phi_0) \left(\frac{\partial R}{\partial \theta_0} d\theta_0 \cos \theta_2 \right). \tag{94}$$

Equation (91) is then

$$L = \sqrt{R \frac{\partial R}{\partial \theta_0} \frac{\cos \theta_\nu}{\sin \theta_0}} \sqrt{\frac{\cos \theta_0}{\cos \theta_\nu}}$$

or

$$L = \sqrt{R \frac{\partial R}{\partial \theta_0} \cot \theta_0} . \tag{95}$$

The solution for R, and therefore $\partial R/\partial \theta_0$, is found by the calculus of variations from Euler's equation in almost every text on mathematical physics

$$R = \int_{z_0}^{z} \frac{pc(z)\, dz}{[1 - p^2 c^2(z)]^{1/2}} \tag{96}$$

where the ray parameter, p, is

$$p = \frac{\sin \theta_0}{c_0} . \tag{97}$$

This completes the set of equations necessary to apply asymptotictray theory to homogeneous layered media. As a simple but instructive example of their application, they will be used in the case of reflected, refracted, and head waves in a two-layered liquid medium.

Amplitude of direct, refracted and reflected waves in liquid media. Consider a point source at Z_0 above an interface and a receiver at point (Z, R) in the same medium. Since $c = \alpha_1$ and $p = \sin \theta_0/\alpha_1$, equation (96) may be easily solved for R and $\partial R/\partial \theta_0$.

$$R = \frac{\alpha_1 p}{(1 - p^2\alpha_1^2)^{1/2}} (Z - Z_0) = (Z - Z_0) \tan \theta_0 \tag{98}$$

$$\frac{\partial R}{\partial \theta_0} = \frac{(Z - Z_0)}{\cos^2 \theta_0}. \tag{99}$$

By (95)

$$L = \frac{Z - Z_0}{\cos \theta_0} = r. \tag{100}$$

For an explosive source in a liquid, the radiation is constant in all directions, therefore, we let $c_0(\theta_0, \phi_0) = 1$. Equation (85) for the amplitude of the direct wave in the first layer is then

$$P_{00} = \frac{1}{r}. \tag{101}$$

For a liquid-liquid interface, the boundary conditions are continuity of vertical displacement and normal stress. Equations (77) and (78) may be written with terms in $\nu = 3, 4$ omitted since shear is not transmitted in a liquid and $\mu_0 = 0$

$$\sum_{\nu=1}^{2} P_{n\nu} \cos \theta_\nu = P_{n0} \cos \theta_0 + \sum_{\nu=0}^{2} (-1)^{\delta_{\nu 0}} p_{n\nu} \sin \theta_\nu \tag{102}$$

$$\sum_{\nu=1}^{2} (-1)^\nu \frac{\lambda_\nu}{\alpha_\nu} P_{n\nu} = \frac{\lambda_0}{\alpha_0} P_{n0} + \sum_{\nu=0}^{2} (-1)^{\nu+\delta_{\nu 0}} [\lambda_\nu \nabla \cdot \bar{P}_{n-1,\nu}] \qquad n = 0, 1, 2, \cdots \tag{103}$$

where $\nu = 0$ indicates an incident P wave, $\nu = 1$ is a reflected P wave and $\nu = 2$ is a transmitted or refracted P wave. For $n = 0$, $\bar{\nabla} \cdot P_{-1\nu} = 0$ by (23) and by (39), $p_{00} = p_{01} = p_{02} = 0$ as the leading term has no components perpendicular to the ray. Setting $\lambda_1 = \rho_1\alpha_1^2$ and $\lambda_2 = \rho_2\alpha_2^2$, equations (102) and (103) are written in matrix form as

$$\begin{pmatrix} \cos \theta_1 & \cos \theta_2 \\ -\rho_1\alpha_1 & \rho_2\alpha_2 \end{pmatrix} \begin{pmatrix} P_{01} \\ P_{02} \end{pmatrix} = \begin{pmatrix} P_{00} & \cos \theta_1 \\ P_{00} & \rho_1\alpha_1 \end{pmatrix}. \tag{104}$$

The solution for the coefficients of reflection, K_1 and refraction, K_2 are readily found to be

$$\frac{P_{01}}{P_{00}} = K_1 = \frac{(\rho_2/\rho_1) \cos \theta_1 - (\alpha_1/\alpha_2) \cos \theta_2}{(\rho_2/\rho_1) \cos \theta_1 + (\alpha_1/\alpha_2) \cos \theta_2} \tag{105}$$

$$\frac{P_{02}}{P_{00}} = K_2 = \frac{2(\alpha_1/\alpha_2) \cos \theta_1}{(\rho_2/\rho_1) \cos \theta_1 + (\alpha_1/\alpha_2) \cos \theta_2}. \tag{106}$$

These are in agreement with equations in Ewing, Jardetzky and Press (1957).

From Figure 3 or by equations (96) and (98), the horizontal position of the wave front for a receiver at depth D in the second medium and its derivative are

$$R = Z_0 \tan \theta_1 + D \tan \theta_2 \tag{107}$$

$$\frac{\partial R}{\partial \theta_0} = \frac{\partial R}{\partial \theta_1} = \frac{Z_0}{\cos^2 \theta_1} + \frac{D}{\cos^2 \theta_2} \frac{\partial \theta_2}{\partial \theta_1} \tag{108}$$

where θ_1 and θ_2 are related by Snell's Law [equation (72)] which upon differentiation becomes

$$\frac{\cos \theta_1}{\alpha_1} d\theta_1 = \frac{\cos \theta_2}{\alpha_2} d\theta_2 . \tag{109}$$

Substituting (108) and (109) into (95) yields the geometric expansion factor

$$L = \cos \theta_1 \sqrt{\frac{R}{\sin \theta_1}} \sqrt{\frac{Z_0}{\cos^3 \theta_1} + \frac{D\alpha_2}{\alpha_1 \cos^3 \theta_2}} . \tag{110}$$

By (90) the amplitude of refracted waves in the second medium is

$$P_{02} = \frac{K_2 \sqrt{\sin \theta_1}}{\cos \theta_1 \sqrt{R} \sqrt{\dfrac{Z_0}{\cos^3 \theta_1} + \dfrac{D\alpha_2}{\alpha_1 \cos^3 \theta_2}}} . \tag{111}$$

For the wave reflected to a receiver at M_1

$$R_1 = (Z_0 + Z_1) \tan \theta_1$$

and

$$L = \sqrt{R_1 \frac{\cos \theta_1}{\sin \theta_1} \frac{(Z_0 + Z_1)}{\cos^2 \theta_1}} = \sqrt{\frac{(Z_0 + Z_1)}{\sin \theta_1} \cdot \frac{R_1}{\cos \theta_1}} = \tilde{R}$$

where $\tilde{R}$ is the distance from the source image to the recording point (see Figure 3). The equation for the amplitude (92) is then

$$P_{01} = \frac{K_1}{\tilde{R}} \tag{112}$$

Amplitude of head waves in a liquid medium. When the angle of incidence, θ_0, is less than the critical angle

$$\theta_0 \leqq \theta_c = \sin^{-1} \frac{\alpha_1}{\alpha_2} \tag{113}$$

equation (74) is real and may be written as

$$\left[\frac{\partial \tau_\nu}{\partial z}\right]_{z=0} = (-1)^{\nu+1} \sqrt{\frac{1}{\alpha_\nu^2} - \left(\frac{\partial \tau_0}{\partial x}\right)^2 - \left(\frac{\partial \tau_0}{\partial y}\right)^2} \quad \nu = 1, 2 \tag{114}$$

because of the requirement of continuity of phase (70) at the boundary.

However, when θ_0 is greater then the critical angle, θ_c,

$$\theta_0 > \sin^{-1} \frac{\alpha_1}{\alpha_2} \tag{115}$$

then by (72) θ_2 is complex and we shall designate it θ_6 as it represents a different type of wave.

$$[\sin \theta_2]_{\theta_0 \geqq \theta_c} = \sin \theta_6 = \frac{\alpha_2}{\alpha_1} \sin \theta_0 . \tag{116}$$

Equation (114) for $\nu = 2$ becomes imaginary and may be written

$$\left[\frac{\partial \tau_2}{\partial z}\right]_{z \to 0} = - i \sqrt{\left(\frac{\partial \tau_0}{\partial x}\right)^2 + \left(\frac{\partial \tau_0}{\partial y}\right)^2 - \frac{1}{\alpha_2^2}} \tag{117}$$

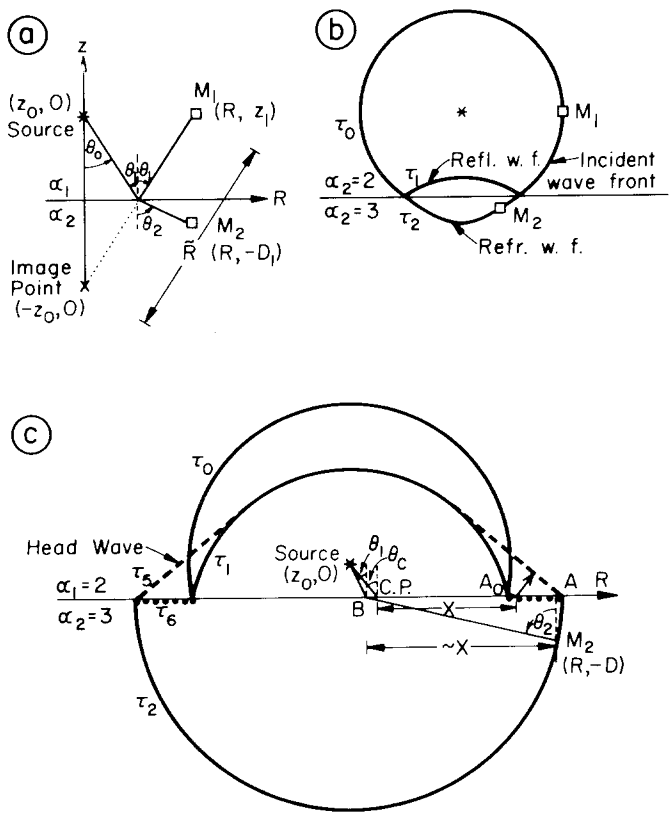

FIG. 3. Diagrams a and b show the rays and wave fronts for waves refracted at less than the critical angle. Diagram c illustrates the rays and wave fronts when the refracted wave front has progressed beyond the critical point, C.P. The same coordinate system is used as in diagram a. If the source in diagrams b and c is 1 km above the interface, diagram a illustrates the case at $t = 0.6$ sec while diagram c is at $t = 2.5$ sec with α_1 and α_2 being 2 and 3 km/sec, respectively.

and complex τ_2 is written as

$$\tau_6 = [\tau_2(x, y, z)]_{z \to 0} = \tau_R(x, y, 0) + i\tau_I(x, y, z) \tag{118}$$

where τ_R is a solution of

$$\text{grad } \tau_2 = \frac{1}{\alpha_2} \tag{119}$$

and from (114) for small negative z

$$\left|\frac{\partial \tau_I}{\partial z}\right|_{z=0} = \sqrt{\left(\frac{\partial \tau_0}{\partial x}\right)^2 + \left(\frac{\partial \tau_0}{\partial y}\right)^2 - \frac{1}{\alpha_2{}^2}}. \tag{120}$$

The imaginary part of τ_6 indicates that an inhomogeneous wave is propagating along the interface. The particle motion is given by substituting in (9)

$$\bar{u}_6(x, y, z, t) = \sum_{n=0}^{\infty} s_n(t - \tau_6) P_{n6}(x, y, z) \tag{121}$$

where by (12)

$$s_n(t - \tau_6) = \frac{1}{\pi} R \int_{\omega_0}^{\infty} \frac{S(\omega)}{(i\omega)^n} e^{\tau_I \omega} e^{i\omega[t - \tau_R(x,y,0)]} \, d\omega. \tag{122}$$

From Figure 3(c)

$$\tau_R = \frac{\sqrt{R^2 + Z_0^2}}{\alpha_1} \tag{123}$$

where R is the horizontal distance from the source pattern to the position of the observer. Using equations (71) to (73) and substituting (113) for α_2 into (120) the solution for τ_I is obtained from

$$\left|\frac{\partial \tau_I}{\partial z}\right| = \sqrt{\frac{\sin^2 \theta_0 - \sin^2 \theta_c}{\alpha_1{}^2}} \tag{124}$$

or

$$\tau_I = \frac{z}{\alpha_1} \sqrt{\sin^2 \theta_0 - \sin^2 \theta_c} \qquad z \leqq 0. \tag{125}$$

The evanescent nature of this wave for which $\nu = 6$ and is otherwise denoted $P_1P_2P_2$ is apparent from (125) and (122). The wave is of little interest in practice since it cannot be observed on the surface. However, a diffracted wave, $\nu = 5$, usually called a $P_1P_2P_1$ head wave, with amplitude denoted by P_{n5}, is generated simultaneously at the interface and does propagate to the surface. The amplitudes of waves 5 and 6 are determined by satisfying the boundary conditions for an incident wave, $\nu = 2$, as it approaches the interface.

Close to the critical angle Snell's Law [equation (72)] may be written as

$$\sin \theta_1 = \frac{\alpha_1}{\alpha_2}, \; \sin \theta_2 \simeq \frac{\alpha_1}{\alpha_2}\left(1 - \delta \frac{\alpha_2}{\alpha_1}\right) \tag{126}$$

where δ is a small quantity to be defined in terms of D, the distance M_2 is below the interface in Figure 3(c). Omitting second order terms

$$\cos \theta_1 \simeq \sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2}; \; \sin \theta_2 \simeq 1; \; \cos \theta_2 \simeq \sqrt{\frac{2\delta\alpha_2}{\alpha_1}}. \tag{127}$$

In terms of the length, X, from the critical point to A, a point on the interface to which the refracted wave, $\nu = 2$, has reached

$$\frac{X}{D} \simeq \tan \theta_2 = \sqrt{\frac{\alpha_1}{2\delta\alpha_2}} \tag{128}$$

or solving for δ

$$\delta = \frac{D^2\alpha_1}{2X^2\alpha_2}. \tag{129}$$

Substitute (126), (127), and (129) into equation (111). The amplitude near point A in layer 2 is then

$$P_{02}(A-) \simeq \frac{K_2 \sqrt{\frac{\alpha_1}{\alpha_2} - \delta}\, \sqrt{\frac{\alpha_1}{\alpha_2}}\, D}{\sqrt{R}\, \sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2}\, X^{3/2}} \tag{130}$$

in which the term in $Z_0/\cos^3 \theta$ has been omitted since it is negligible in comparison to X^3/D^2. In the limit

$$\lim_{D\to 0} P_{02}(M_2) = P_{02}(A) = 0. \tag{131}$$

However, the derivative is not zero

$$\lim_{D\to 0} \frac{\partial P_{02}(M_2)}{\partial z} = \frac{\partial P_{02}(A)}{\partial z} = \frac{-\left(\frac{\alpha_1}{\alpha_2}\right) K_2}{\sqrt{R}\, X^{3/2} \sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2}}. \tag{132}$$

In accordance with (9) the solutions for the evanescent wave ($\nu = 6$) and the head wave ($\nu = 5$) generated at poinsts like A will be

$$\bar{u}_\nu(x, y, z, t) = \sum_{n=0}^{\infty} \bar{P}_{n\nu}(x, y, z) s_n(t - \tau_\nu(x, y, z)) \qquad \nu = 5, 6. \tag{133}$$

This solution may be substituted into the boundary conditions given by equations (102) and (103) and by letting

$$\nu = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}_{\theta_0 < \theta_c} = \begin{Bmatrix} 2 \\ 5 \\ 6 \end{Bmatrix}_{\theta_0 \geqq \theta_c}. \tag{134}$$

Continuity of vertical displacement is then

$$\sum_{\nu=5}^{6} P_{n\nu} \cos \theta_\nu = \lim_{D\to 0} \left\{ P_{n2} \cos \theta_2 - p_{n2} \sin \theta_2 + \sum_{\nu=5}^{6} p_{n\nu} \sin \theta_\nu \right\} \quad n = 0, 1, 2 \cdots. \tag{135}$$

Continuity of stress is

$$\sum_{\nu=5}^{6} (-1)^{\nu} \frac{\lambda_{\nu-4}}{\alpha_{\nu-4}} P_{n\nu} = \lim_{D\to 0} \left\{ \frac{\lambda_2}{\alpha_2} P_{n2} - \lambda_2 \nabla \cdot \bar{P}_{n-1,2} + \sum_{\nu=5}^{6} (-1)^{\nu} [\lambda_{\nu-4} \nabla \cdot \bar{P}_{n-1,\nu}] \right\}. \quad (136)$$

Setting $n = 0$ and noting that by (23) and (39) $P_{-1\nu} = p_{00} = p_{01} = p_{02} = 0$ we have

$$\begin{pmatrix} \cos\theta_5 & \cos\theta_6 \\ \dfrac{-\lambda_1}{\alpha_1} & \dfrac{\lambda_2}{\alpha_2} \end{pmatrix} \begin{pmatrix} P_{05} \\ P_{06} \end{pmatrix} = \lim_{D\to 0} \begin{pmatrix} P_{02}\cos\theta_2 \\ \dfrac{\lambda_2}{\alpha_2} P_{02} \end{pmatrix} \quad (137)$$

but by equation (131) the right-hand side is identically zero. Therefore, for $n = 0$ the only solution is

$$P_{0\nu} \equiv 0 \qquad \nu = 5, 6. \quad (138)$$

Such a result was anticipated since the incident wave had a leading term which is zero on the boundary.

Next set $n = 1$ in equation (135) and (136).

$$\begin{pmatrix} \cos\theta_5 & \cos\theta_6 \\ -\lambda_1/\alpha_1 & \lambda_2/\alpha_2 \end{pmatrix} \begin{pmatrix} P_{15} \\ P_{16} \end{pmatrix} = \lim_{D\to 0} \begin{pmatrix} P_{12}\cos\theta_2 - p_{12}\sin\theta_2 + \sum_{\nu=5}^{6} p_{1\nu}\sin\theta_\nu \\ \dfrac{\lambda_2}{\alpha_2} P_{12} - \lambda_2 \nabla \cdot \bar{P}_{02} + \sum_{\nu=5}^{6} (-1)^{\nu} \lambda_{\nu-4} \nabla \cdot \bar{P}_{0\nu} \end{pmatrix}. \quad (139)$$

From (138) and (52), it is obvious that $p_{15} = p_{16} = 0$ and, therefore, the only quantity on the right-hand side that needs to be determined is p_{12}. From (52)

$$p_{12} = \frac{-\alpha_2^{\,2}}{\lambda_2} [\bar{M}(\tau_{\alpha 2}, \bar{P}_{02})] \cdot \bar{n} \quad (140)$$

where $\bar{n}$ is a unit vector in the vertical (z) direction and grad $\tau_{\alpha 2}$ is a vector ($\bar{r}/\alpha_2$) along the interface. Substituting (20) with $\mu = 0$

$$p_{12} = -\alpha_2^{\,2} [(\nabla \cdot \bar{P}_{02}) \operatorname{grad} \tau_{\alpha 2} + \operatorname{grad} (P_{02} \cdot \operatorname{grad} \tau_{\alpha_2})] \cdot \bar{n}. \quad (141)$$

In the first term grad $\tau_{\alpha 2} \cdot \bar{n}$ is zero and the second term is $\bar{P}_{02} \cdot \operatorname{grad} \tau_{\alpha 2} = [P_{02}\bar{r} + p_{02}\bar{n}) \cdot \bar{r}/\alpha_2 = P_{02}/\alpha_2$. Taking the gradient operator in (141) in circular cylindrical coordinates, it is found that the only nonzero component when projected in the $\bar{n}$ direction is (with $\bar{n} \cdot \bar{k} = -1$)

$$p_{12} = \lim_{D\to 0} \alpha_2 \frac{\partial P_{02}}{\partial z}. \quad (142)$$

The boundary condition of (139) may now be written with $\nabla \cdot \bar{P}_{02} = 0$ in the limit as D approaches zero and also with $p_{12} = p_{15} = \bar{P}_{05} = \bar{P}_{06} = \nabla \cdot \bar{P}_{05} = \nabla \cdot \bar{P}_{06} = 0$

$$\begin{pmatrix} \cos\theta_5 & \cos\theta_6 \\ \dfrac{-\lambda_1}{\alpha_1} & \dfrac{\lambda_2}{\alpha_2} \end{pmatrix} \begin{pmatrix} P_{15} \\ P_{16} \end{pmatrix} = \lim_{D\to 0} \begin{pmatrix} P_{12}\cos\theta_2 - \alpha_2 \dfrac{\partial P_{02}}{\partial z} \sin\theta_2 \\ \dfrac{\lambda_2}{\alpha_2} P_{12} \end{pmatrix}. \quad (143)$$

Setting $\theta_2 = \theta_6 = 90°$ and $\sin\theta_5 = \alpha_1/\alpha_2$, the determinant of the coefficients is

$$\mathrm{Det} = \frac{\lambda_2}{\alpha_2}\cos\theta_5 = \rho_2\alpha_2\sqrt{1-\left(\frac{\alpha_1}{\alpha_2}\right)^2}. \tag{144}$$

The solution for the amplitude of the evanescent wave is

$$P_{16} = \frac{1}{\mathrm{Det}}\begin{vmatrix} \cos\theta_5 & -\lim\limits_{D\to 0}\alpha_2\dfrac{\partial P_{02}}{\partial z} \\ -\alpha_1\rho_1 & \lim\limits_{D\to 0}\rho_2\alpha_2 P_{12} \end{vmatrix}$$

$$= P_{12} + \frac{2\alpha_1^3\rho_1^2}{\rho_2^2\alpha_2^2[1-(\alpha_1/\alpha_2)^2]R^{1/2}X^{3/2}}. \tag{145}$$

The amplitude of the head wave is

$$P_{15} = \frac{1}{\mathrm{Det}}\begin{vmatrix} -\lim\limits_{D\to 0}\alpha_2\dfrac{\partial P_{02}}{\partial z} & 0 \\ \lim\limits_{D\to 0}\rho_2\alpha_2 P_{12} & \rho_2\alpha_2 \end{vmatrix}$$

$$= \frac{+\alpha_1 K_2}{\left[1-\left(\dfrac{\alpha_1}{\alpha_2}\right)^2\right]R^{1/2}X^{3/2}} = \frac{+2\alpha_1^2\rho_1}{\alpha_2\rho_2\left[1-\left(\dfrac{\alpha_1}{\alpha_2}\right)^2\right]R^{1/2}X^{3/2}}. \tag{146}$$

The geometric expansion factor, (95), may be calculated, but equation (146) is not modified if R is interpreted as the horizontal distance from the source to the receiver. The amplitude is identical to that calculated by Brekhovskikh (1960), but the present derivation does not employ contour integration in a complex plane. Coefficients for head waves in elastic solids are most easily obtained following the method outlined in Appendix 2.

Synthetic seismograms. In a medium consisting of $I + 1$ layers, including a half-space, there are an infinite number of rays which propagate from an arbitrarily-located source to a receiver at some other position. Let P be a linear space of all possible phases (i.e., rays) where each phase, $p_1(\bar{x}, t)$, $p_2(\bar{x}, t)$, $\cdots$, is a function of position vector $\bar{x}$ and time. The vast majority of the phases will not contribute measurable energy to any receiver location and these are not of practical interest. Some rays may travel along different paths from the source to the receiver but with identical travel-time curves and these are called kinematic analogs. Groups of such kinematically equivalent phases may contribute a significant component of motion to the synthetic seismogram even though the individual analogs do not do so. The n groups of functions which contribute significantly are a subset of the linear space P. It is the object of a paper by Hron (1971) to classify phases into kinematic analogs and a subgroup of dynamic analogs whose amplitude-distance curves are also identical. A set of rules, R, must then be defined by which rays which have a geophysical interest are retained.

In a homogeneous layered medium, let us describe the top of the first layer as the zeroth interface. As each ray travels in a straight line through each layer, it may be subdivided into $k = 1$ to K connected segments. Since each segment is a result of a reflection, refraction or diffraction in which it suffers a loss of energy, the first rule one can establish is an empirical one on the number of segments which will be considered. The second empirical rule sets a minimum amplitude level as a function of time which a set of dynamic analogs must attain to be incorporated in the seismogram. The combination of these two rules is used to limit the requirements on the size of the memory and the computation time.

As an example, consider a three-layer case in which the program is required to calculate all phases with 2 to 8 segments when both source and receiver are on the surface. All dynamic analogs and their amplitude-distance curve at selected distances are obtained for $K = 2, 4, 6$, and 8 segments of unconverted phases. The results for $K = 8$, which are summarized in Figure 4, are obtained as follows. From Hron's (1971), equation 11, the number of kinematic analogs for J layers above a half-space and $\frac{1}{2}n_i$ elements in each layer i is

$$N_k(n_1, \cdots, n_J) = \prod_{j=1}^{J-1} C_{n_j+1}^{n_j+n_{j+1}-1} \tag{147}$$

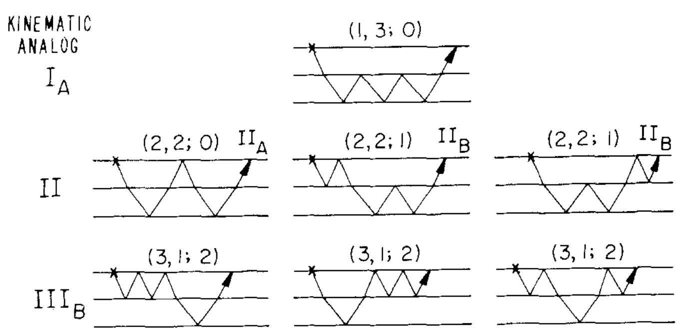

FIG. 4. Kinematic and dynamic analogs of multiply-reflected P waves in a three-layer case for eight segments in the ray. The three phases in analog I arrive at the same time but the amplitude of analog II_A is different from II_B. All analogs labelled III arrive at the same time and with the same amplitude.

where

$$2 \sum_{i=1}^{J-1} n_i = K. \tag{148}$$

Condition (148) is satisfied for three kinematic analogs: $N_I(1, 3) = 1$; $N_{II}(2, 2) = 3$ and $N_{III}(3, 1) = 3$. The dynamic analogs of these seven phases are determined from Hron's equation 14 which is the condition on the number reflections, m_j, from above off the jth interface,

$$\text{Max } (0, n_j - n_{j+1}) \leqq m_j \leqq n_{j-1} \qquad 1 \leqq j \leqq J - 1 \tag{149}$$

and the number of dynamic analogs

$$N_d(n_1, \cdots, n_J; m, \cdots, m_{J-1}) = \prod_{j=1}^{J-1} C_{m_{n_j}}^{n_j} C_{n_j-m_j-1}^{n_{J+1}-1}. \tag{150}$$

There are two dynamic analogs for phases (2, 2) since $N_d(2, 2; 0) = 1$ and $N_d(2, 2; 1) = 2$. All three phases in group III have the same amplitude characteristics so their reflection coefficient needs to be calculated only once. A similar analysis can be followed for the converted phases and the head waves.

Asymptotic ray theory, as outlined in the preceding sections, has been applied to a plane multilayered homogeneous elastic medium and a point source. The algorithms were programmed using a FORTRAN IV language and applied on an IBM 360-67 computer. The display was made using a Calcomp plotter.

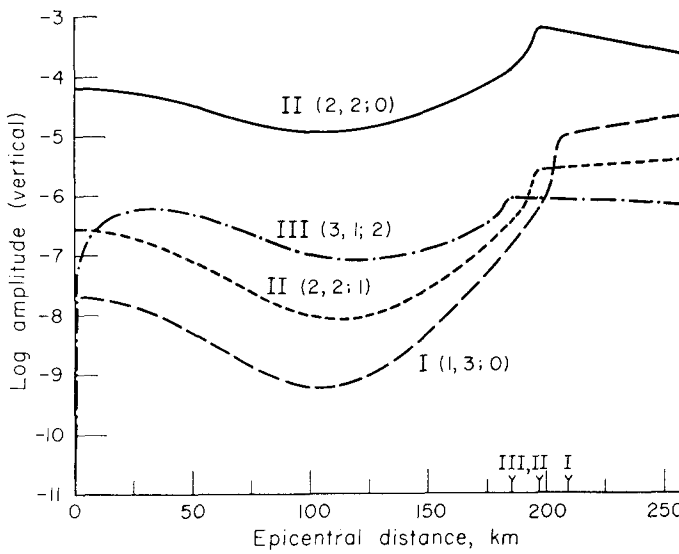

FIG. 5. Amplitude-distance characteristics for the analogs in Figure 4 for a northwestern Ontario model shown in Figure 7.

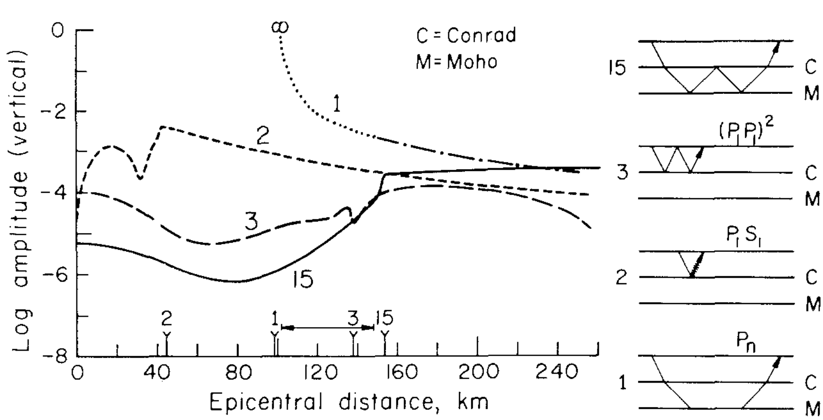

FIG. 6. Amplitude-distance characteristics for the vertical component of four different phases found in the northwestern Ontario model. The key to the wave numbers is in the *upper right-hand insert*. The points, Y, on the distance scale indicate critical distances. The *dotted portion* of the head wave was not used in computing synthetic seismograms.

The first model for which computations were carried out is a three-layer case typical of northwestern Ontario and southeastern Manitoba as proposed by Hall and Hajnal (1969) from deep seismic sounding studies. The impulse response was chosen to be

$$s(t) = \sin(\omega_0 t) \exp(-(\omega_0 t/\gamma)^2) \tag{151}$$

$$S(\omega) = \frac{-i\pi^{1/2}\gamma}{\omega_0} \exp\left(-\frac{\gamma^2}{4}\left(1+\left(\frac{\omega}{\omega_0}\right)^2\right)\right) \sinh\left(\frac{\gamma^2}{2}\frac{\omega}{\omega_0}\right) \tag{151a}$$

where

$$\omega_0 = 2\pi f_0$$

and $f_0 = 2.5\ Hz$, $\gamma = 2.0$. Figure 5 shows the amplitude characteristics of the dynamic

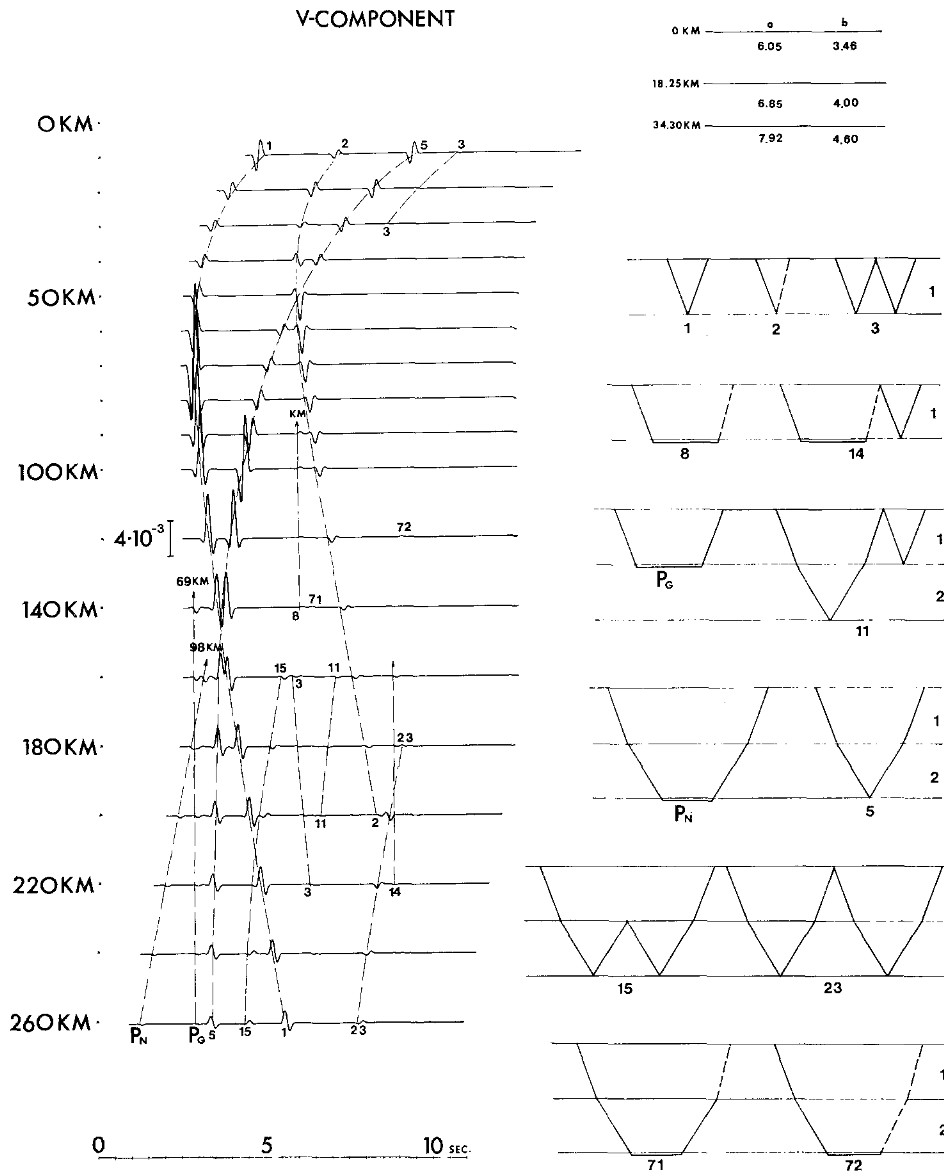

FIG. 7. Synthetic seismograms for the vertical component plotted with a reduced velocity of 6.85 km/sec for the model in the *upper right corner*. The phase identification is on the *right*.

analogs presented in Figure 4, whereas Figure 6 compares the reflection and head-wave coefficients for a number of other phases. It is seen that the amplitudes of several multiply-reflected and converted phases are comparable to the head-wave amplitude.

The synthetic seismograms and the northwestern Ontario model are shown in Figure 7. Only the vertical component is displayed although the horizontal one is available. The arrivals are plotted as a reduced section ($T - R/6.85$ km/sec) between distances

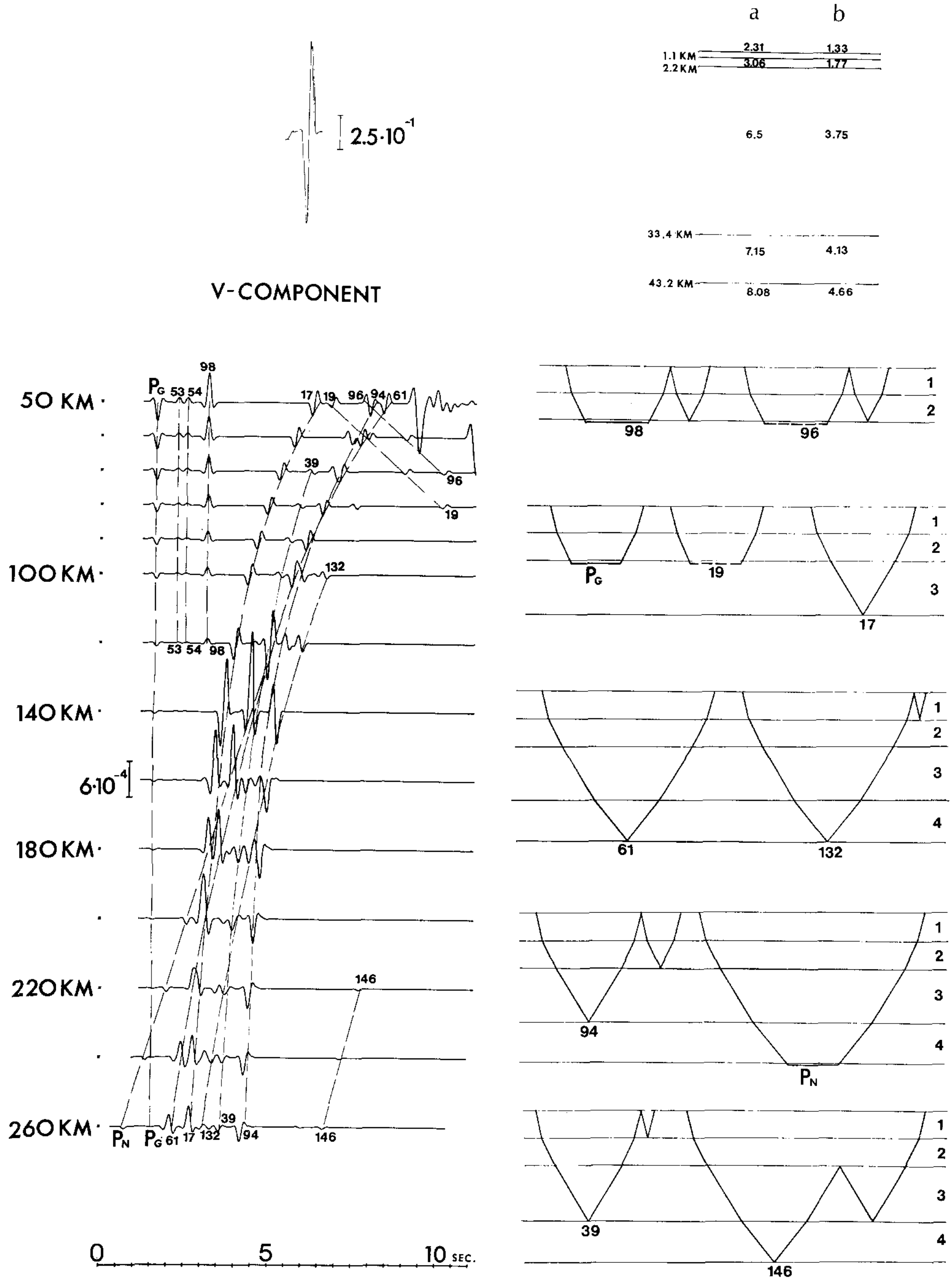

FIG. 8. Synthetic seismograms for the vertical component plotted with a reduced velocity of 6.5 km/sec for the Alberta model. Phase identification is given on the *right*.

of 10 and 260 km. It should be noted that the direct wave has been omitted as it has a very large amplitude which makes scaling difficult. In field experiments, the direct wave is rather weak due to scattering by near-surface inhomogeneities. The P_n, P_g, and $P_1P_2S_1$ head waves are also omitted between the critical point and the distance indicated on the seismogram since our first-order asymptotic expansion is not valid in this region. Up to 12 segments and 174 phases were considered.

The second group of seismograms was calculated for a five-layer southern Alberta model as determined by studies of Cumming and Kanasewich (1966). The same pulse was used as in the preceding section. A maximum of 10 segments was allowed and a total of 310 phases was considered for inclusion in the seismogram. The wave symbols and the dynamic analogs are identified in the Appendix 3. Considerably greater complication is introduced by adding the two additional layers near the surface. In Figure 8 and 9 for the vertical and horizontal components plotted at a reduced velocity of 6.50 km/sec, the dominance of reflections over head waves is very noticeable. Some rather exotic phases such as 96, 98, and 146 have surprisingly large amplitudes. In Figure 10,

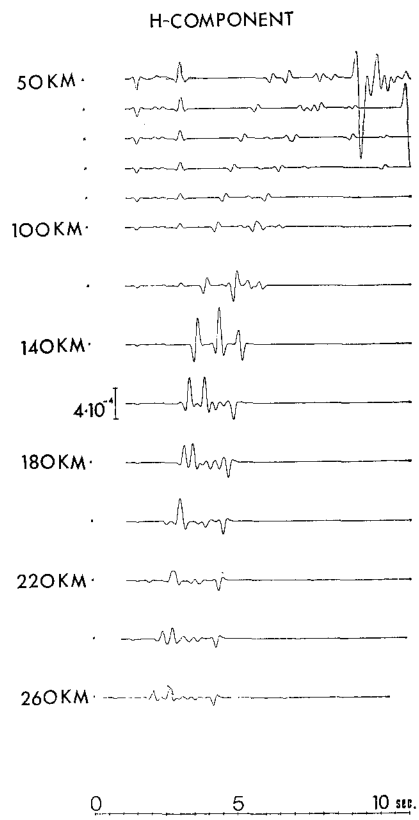

FIG. 9. Synthetic seismograms for the horizontal component plotted with a reduced velocity of 6.5 km/sec for the southern Alberta model. Phase identification may be made from Figure 8.

the later phases are included to show the dominance of such events as P_1P_1 and $P_1(P_2P_2)^2P_1$ at large distances. At short distances, the seismograms become very complicated when there are thin layers close to the surface. This is very apparent in Figure 11 which is a seismogram for an Alberta model at a distance of 20 km. This is the first output from the Calcomp plotter, and one can see the system used to indicate the wave identification number, its time of arrival, and the amplitude to one significant figure. Unless there is a substantial proportion of high-frequency energy in the source spectrum, it becomes impossible to resolve the events at these distances.

From the examples presented here, it appears that reflected phases are very dominant immediately after the first head-wave arrival. Indeed, events such as 17, 61 or 94 in Figure 8 may easily be misinterpreted as late-arriving head waves. Červený (1966) has noted that head waves must rarely exist because of the effects of a positive

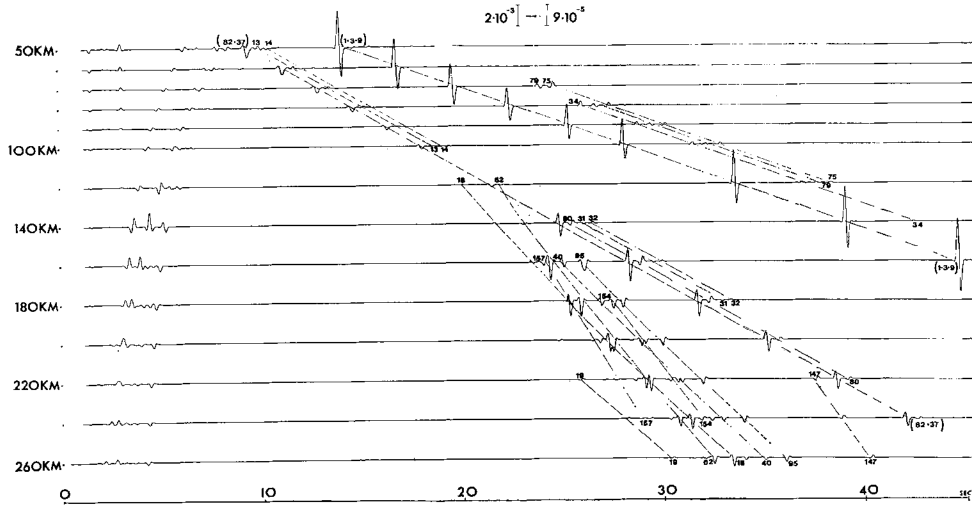

FIG. 10. Synthetic seismograms for the vertical component plotted with a reduced velocity of 6.5 km/sec for the southern Alberta model. The amplitude is reduced to one-third of that in Figure 8. The phases are identified in Appendix 3.

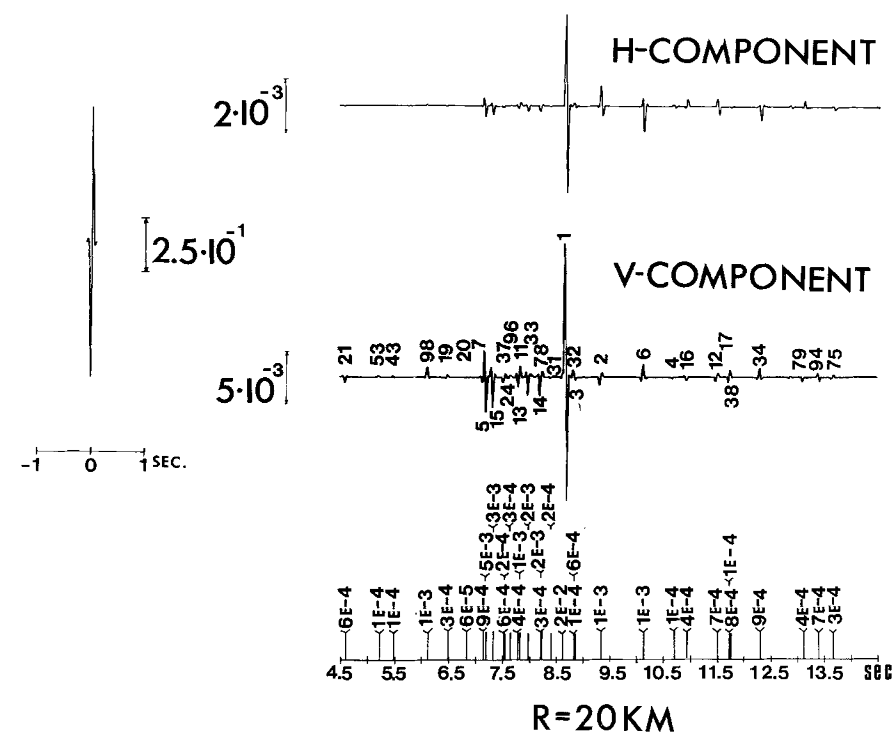

FIG. 11. Synthetic seismogram at 20 km for the Alberta model showing both vertical and horizontal components. The phases are identified by wave number above the seismogram and keyed in Appendix 3. The relative amplitude of each phase is given above the time scale as a floating point number (6E-4 $\equiv 6 \times 10^{-4}$). The mid-frequency of the source pulse was 10 Hz in equation (151) for the figure as compared to 2.5 Hz for all other figures.

velocity gradient in a medium and also because of the curvature of the Earth. Thus, we conclude that phases should be examined on the basis that they may be reflections—multiple or converted reflecting or plunging reflections from a layer with velocity increasing with depth. Seismograms computed from generalized ray theory appear to be very useful because they contain not only the time of arrival and the intensities of the waves but also a positive identification of each of the phases.

Acknowledgments

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Appendix 1

Gauss's theorem may be applied to the flux of rays in a small element of a ray tube (see *insert* in Figure 1)

$$\iiint_v \nabla^2\tau \, dV = \iint_A \text{grad } \tau \cdot d\bar{A}.$$

The components of the gradient are found by applying conservation of energy so that the flux, $\partial\tau/\partial r_3$ through the sides of the ray tube is zero and by (32)

$$\frac{\partial\tau}{\partial r_1} = \frac{-1}{\alpha_1}; \frac{\partial\tau}{\partial r_2} = \frac{1}{\alpha_2}.$$

$$\iiint_V \nabla^2\tau \, dV = \iint_A - \left[\frac{dA_1}{\alpha_1} - \frac{dA_2}{\alpha_2}\right] = \iint_A - \left[\frac{J_1}{\alpha_1} - \frac{J_2}{\alpha_2}\right] du_2 du_3$$

where u_2 and u_3 are curvilinear coordinates specifying the central ray of the ray tube and J is the divergence of the ray tube [equation (36)]. Then

$$\iiint_V \nabla^2\tau \, dV = \iint_A \partial\left(\frac{J}{\alpha}\right) du_2 \, du_3 = \iiint_{\tau_1}^{\tau_2} \frac{\partial(J/\alpha)}{J\alpha \, d\tau} J \, du_2 \, du_3 \, \alpha d\tau$$

where $\alpha d\tau$ is the length of the ray tube. Since $dV = J \, du_2 \, du_3 \, \alpha d\tau$, we have that the divergence of the ray tube is given by

$$\nabla^2\tau = \nabla \cdot \text{grad } \tau = \frac{1}{J\alpha} \frac{d}{d\tau}\left(\frac{J}{\alpha}\right). \tag{A1}$$

Also grad $\tau \cdot \nabla$ is a directional derivative along the ray which can be written in terms of a differential, dS, along the ray.

$$\text{grad } \tau \cdot \nabla = \frac{1}{\alpha} \frac{d}{dS} = \frac{1}{\alpha^2} \frac{d}{d\tau}. \tag{A2}$$

The operator $\bar{M}$ in equation (20) can be reduced to a simpler form when applied to the coefficient in the direction of the ray.

$$\bar{M}(\tau_\alpha, P_n \text{ grad } \tau_\alpha) = (\lambda + \mu)[\nabla \cdot (P_n \text{ grad } \tau_\alpha) \text{ grad } \tau_\alpha + \text{grad } (P_n \text{ grad } \tau_\alpha \cdot \text{grad } \tau_\alpha)] + \mu[(P_n \text{ grad } \tau_\alpha)\nabla^2\tau_\alpha + I] \tag{A3}$$

where

$$I = 2\sum_{j=1}^{3} \bar{e}_j \text{ grad } \tau_\alpha \cdot \text{grad}\left(P_n \frac{\partial \tau_\alpha}{\partial x_j}\right)$$

$$= 2\left\{\bar{\imath} \text{ grad } \tau_\alpha \text{ grad}\left(P_n \frac{\partial \tau_\alpha}{\partial x_1}\right) + \bar{\jmath} \text{ grad } \tau_\alpha \cdot \text{grad}\left(P_n \frac{\partial \tau_\alpha}{\partial x_2}\right) + \bar{k} \text{ grad } \tau_\alpha \cdot \text{grad}\left(P_n \frac{\partial \tau_\alpha}{\partial x_3}\right)\right\}$$

$$I = 2\{(\text{grad } P_n \cdot \text{grad } \tau_\alpha) \text{ grad } \tau_\alpha\} + 2P_n\left\{\bar{\imath} \text{ grad } \tau_\alpha \cdot \text{grad } \frac{\partial \tau_\alpha}{\partial x_1} + \bar{\jmath} \text{ grad } \tau_\alpha \text{ grad } \frac{\partial \tau_\alpha}{\partial x_2} + \bar{k} \text{ grad } \tau_\alpha \text{ grad } \frac{\partial \tau_\alpha}{\partial x_3}\right\}$$

$$= 2(\text{grad } P_n \cdot \text{grad } \tau_\alpha) \text{ grad } \tau_\alpha \tag{A4}$$

since the term in braces is grad (grad $\tau_\alpha)^2$ = grad $(1/\alpha^2)$ which is zero for homogeneous media.

$$\bar{M} = (\lambda + \mu)[(\text{grad } P_n \cdot \text{grad } \tau_\alpha) \text{ grad } \tau_\alpha + (P_n \nabla^2 \tau_\alpha) \text{ grad } \tau_\alpha + \text{grad } P_n (\text{grad } \tau_\alpha)^2] + \mu P_n \nabla^2 \tau_\alpha \text{ grad } \tau_\alpha + 2\mu (\text{grad } P_n \cdot \text{grad } \tau_\alpha) \text{ grad } \tau_\alpha . \tag{A5}$$

$$\bar{M} = \text{grad } \tau_\alpha [(\lambda + 2\mu) \text{ grad } P_n \cdot \text{grad } \tau_\alpha + \mu \text{ grad } P_n \cdot \text{grad } \tau_\alpha + (\lambda + 2\mu) P_n \nabla^2 \tau_\alpha] + (\lambda + \mu) \text{ grad}\left(\frac{P_n}{\alpha^2}\right). \tag{A6}$$

Projecting $\bar{M}$ in the direction grad τ_α gives the following in which we multiply and divide by ρ, note that (grad $\tau_\alpha)^2 = 1/\alpha^2$ and add and subtract a quantity in μ grad P_n to the last term

$$\bar{M}(\tau_\alpha, P_n \text{ grad } \tau_\alpha) \cdot \text{grad } \tau_\alpha = \frac{\rho}{\alpha^2}\left[\alpha^2 (\text{grad } P_n \cdot \text{grad } \tau_\alpha) + \frac{\mu}{\rho} \text{ grad } P_n \cdot \text{grad } \tau_\alpha + \alpha^2 P_n \nabla^2 \tau_\alpha\right] + \rho \text{ grad } \tau_\alpha \cdot \left[\frac{\lambda + 2\mu}{\alpha^2 \rho} \text{ grad } P_n - \frac{\mu}{\alpha^2 \rho} \text{ grad } P_n\right]$$

$$\bar{M} \cdot \text{grad } \tau_\alpha = \frac{\rho}{\alpha^2}\{\alpha^2 \text{ grad } P_n \cdot \text{grad } \tau_\alpha + \alpha^2 P_n \nabla^2 \tau_\alpha + \alpha^2 \text{ grad } \tau_\alpha \cdot \text{grad } P_n\}. \tag{A7}$$

Substituting (A1) and (A2) yields

$$\bar{M} \cdot \text{grad } \tau_\alpha = \frac{\rho}{\alpha^2}\left\{2 \frac{dP_n}{d\tau_\alpha} + \frac{P_n}{J} \frac{dJ}{d\tau_\alpha}\right\} \tag{A8}$$

or more compactly as

$$\bar{M}(\tau_\alpha, P_n \text{ grad } \tau_\alpha) \cdot \text{grad } \tau_\alpha = \frac{2\rho}{\alpha^2 J^{1/2}} \frac{d}{d\tau_\alpha} (P_n J^{1/2}) . \tag{A9}$$

Appendix 2

HEAD WAVES

The leading term in the expansion for head waves, of which equation (146) is an example, may be obtained following a method due to Petrashen (1957, 1959) and Podyapolski (1959). From Petrashen (1957), the general equation for the complex amplitude of the leading term of any head waves in (7) for a two-layered medium is

$$A = \frac{C_{q_1} \Gamma_{q_1 q_H q_2}}{\cos \theta_c \cdot R^{1/2} X^{3/2}} \tag{A10}$$

where X is the distance along the interface, R is epicentral distance, θ_c is the critical angle, c is the phase velocity of the incident wave, Γ is the coefficient of the head wave and q_i expresses the wave type ($q = P$ or S; $q_1 =$ incident wave, $q_H =$ wave along interface; $q_2 =$ wave propagating away from interface.)

Following either Petrashen (1957) or Podyapolski (1959), the coefficient for head waves, Γ, may be obtained from the coefficient of reflection, $K_{q_1 q_2}$. The coefficient of reflection is written in terms of functions which are independent of $\cos \theta_{q_H}$ where

$$\cos \theta_{q_H} = \sqrt{1 - (pc_{q_H})^2} \tag{A11}$$

and

$$p = \frac{\sin \theta}{C_{q_1}} = \frac{\sin \theta_{q_H}}{C_{q_H}} \tag{A12}$$

is the ray parameter

$$K_{q_1 q_2}(\theta) = \frac{N_{q_1 q_2}(\theta) + \tilde{N}_{q_2 q_2}(\theta) \cos \theta_{q_H}}{D_{q_1 q_2}(\theta) + \tilde{D}_{q_1 q_2}(\theta) \cos \theta_{q_H}} \tag{A13}$$

multiplying and dividing by $D_{q_1 q_2} - \tilde{D}_{q_1 q_2} \cos \theta_{q_H}$ yields

$$K_{q_1 q_2}(\theta) = \frac{[N_{q_1 q_2} D_{q_1 q_2} - (\cos^2 \theta_{q_H}) \cdot \tilde{N}_{q_1 q_2} \tilde{D}_{q_1 q_2}] - [N_{q_1 q_2} \tilde{D}_{q_1 q_2} - \tilde{N}_{q_1 q_2} D_{q_1 q_2}] \cos \theta_{q_H}}{D^2_{q_1 q_2} - \tilde{D}^2_{q_1 q_2} \cos^2 \theta_{q_H}} .$$

The function in the second pair of square brackets is equal to $N_{j_1 q_1 j_2 q_2 \Gamma}$ in Podyapolski's equation (10). For a head wave $p = 1/c_{q_H}$ and $\cos \theta_{q_H} = 0$, so that, by Podyapolski's equation (9), the coefficient for head waves is equal to the function in the second pair of square brackets

$$\Gamma_{q_1 q_H q_2} = \left[\frac{N_{q_1 q_2} \tilde{D}_{q_1 q_2} - \tilde{N}_{q_1 q_2} D_{q_1 q_2}}{D^2_{q_1 q_2}}\right]_{\theta = \theta_c} . \tag{A15}$$

For a liquid medium, equation (105) yields

$$N_{P_1 P_1} = (\rho_2/\rho_1) \cos \theta_c ; \qquad \tilde{N}_{P_1 P_1} = -(\alpha_1/\alpha_2)$$

$$D_{P_1 P_1} = (\rho_2/\rho_1) \cos \theta_c ; \qquad \tilde{D}_{P_1 P_1} = \alpha_1/\alpha_2$$

and

$$\Gamma_{P_1P_2P_1} = \frac{2\rho_1 \alpha_1}{\rho_2 \alpha_2 \cos \theta_c}. \tag{A16}$$

Then from (A10)

$$A_1 = p_{15} = \frac{\alpha_1^{\ 2} \rho_1}{\alpha_2 \rho_2 \left[1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2\right] R^{1/2} X^{3/2}} \tag{A17}$$

which agrees with equation (146).

For elastic solids equation (A15) is used with the reflection coefficients

$$K_{P_1P_1} \text{ to obtain } \Gamma_{P_1P_2P_1} \text{ and } \Gamma_{P_1S_2P_1},$$

$$K_{P_1S_1} \text{ to obtain } \Gamma_{P_1P_2S_1} \text{ and } \Gamma_{P_1S_2S_1},$$

$$K_{S_1S_1} \text{ to obtain } \Gamma_{S_1P_2S_1} \text{ and } \Gamma_{S_1S_2S_1},$$

and

$$K_{S_1P_1} \text{ to obtain } \Gamma_{S_1P_2P_1} \text{ and } \Gamma_{S_1S_2P_1}.$$

Explicit expressions are given by Podyapolski (1959) for the leading term in the expansion for head waves.

Appendix 3

IDENTIFICATION OF PHASES IN FIGURES 10 AND 11 FOR THE ALBERTA MODEL

Wave Number	Dynamic Analogs	Typical phase	Wave Number	Dynamic Analogs	Typical Phase
1	1	P_1P_1	33	1	$(P_1P_2P_2P_1)^2$
2	1	P_1S_1	34	1	$P_1S_2{}^2S_1{}^2S_2{}^2S_1$
3	1	$(P_1P_1)^2$	37	1	$P_1(P_2P_2)^3P_1$
4	1	$P_1S_1{}^3$	38	1	$P_1(S_2S_2)^3S_1$
5	1	$P_1P_2P_2P_1$	43	2	$P_1P_2P_3P_2P_1{}^3$
6	1	$P_1S_2S_2S_1$	53	2	$P_1P_2P_3P_2{}^3P_1$
7	1	$P_1P_2P_1$	62	1	$P_1S_2S_3S_4{}^2S_3S_2S_1$
8	1	$P_1P_2S_1$	75	2	$P_1S_2{}^2S_1S_1S_2{}^2S_1{}^3$
11	2	$P_1P_2{}^2P_1{}^3$	78	2	$P_1P_2{}^2P_1{}^2(P_2P_2)^2P_1$
12	1	$P_1S_2{}^2S_1{}^3$	79	2	$P_1S_2{}^2S_1(S_2S_2)^2S_1$
13	2	$P_1P_2P_1{}^3$	94	2	$P_1P_2P_3{}^2P_2P_1{}^2P_2{}^2P_1$
14	2	$P_1P_2S_1P_1{}^2$	95	2	$P_1S_2S_3{}^2S_2S_1{}^2S_2{}^2S_1$
15	1	$P_1(P_2P_2)^2P_1$	96	2	$P_1P_2S_3P_2P_1{}^2P_2{}^2P_1$
16	1	$P_1(S_2S_2)^2S_1$	98	2	$P_1P_2P_3P_2P_1{}^2P_2{}^2P_1$
17	1	$P_1P_2P_3{}^2P_2P_1$	147	2	$P_1S_2S_3S_4{}^2S_3{}^3S_2S_1$
18	1	$P_1S_2S_3{}^2S_2S_1$	154	1	$P_1S_2S_3S_4{}^4S_3S_2S_1$
19	1	$P_1P_2S_3P_2P_1$	157	1	$P_1S_2S_3S_4S_5S_4S_3S_2S_1$
20	1	$P_1P_2S_3P_2P_1$	9	1	$(P_1P_2)^3$
21	1	$P_1P_2P_3P_2P_1$	54	1	$P_1P_2P_3S_2P_2{}^2P_1$
24	1	$P_1S_2S_3S_2S_1$	80	2	$P_1P_2{}^6P_1{}^3$
31	3	$P_1P_2P_1(P_1P_1)^2$	82	1	$P_1P_2{}^8P_1$
32	3	$P_1P_2S_1(P_1P_1)^2$			

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